EFFECT OF HEAT EXCHANGE AT THE HOT JUNCTION OF A THERMOELECTRIC ELEMENT ON THE OPTIMUM CURRENT-CARRIER DENSITY DISTRIBUTION ALONG ITS BRANCHES

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The optimum current-carrier density distribution along the height of the branches of a thermoelectric element is obtained, which ensures minimum temperature at the cold junction taking into account the heat transfer at the hot junction.

Investigations have recently been carried out to estimate the effect of the current-carrier density distribution along the arms of a semiconductor thermoelectric element on the efficiency of cooling devices [1-3].

Analysis of the operation of a nonuniform thermoelectric element when the cold junction is adiabatically insulated, when the hot junction is at constant temperature, has enabled the optimum change in the properties of the material along its length to give the maximum possible reduction in temperature at the cold junction to be obtained [4]. The physical parameters of the thermoelectric material was assumed to depend only on the current-carrier density and not on the temperature. This simplification was justified by the fact that in the materials used, which are intended for thermoelectric refrigerators (triple alloys based on Bi_2Te_3), the physical properties vary only slightly over the operating temperature range. It was found that the maximum temperature drop with respect to the medium in the nonuniform thermoelectric element can be increased by 18°K compared with a thermoelectric element made of a uniform material with an optimum current-carrier density [4].

The formulation of the problem in a more general form, taking into account the finite heat transfer at the hot side, should enable the temperature-reduction advantage obtainable, compared with a uniform thermoelectric element, to be estimated for actual conditions and with a different degree of intensification of the heat exchange between the hot function and the medium.

We will assume that both branches of the thermoelectric element, along which the heat propagates, are identical in their physical properties, but the thermoelectric coefficients are of opposite sign. The equation describing the temperature distribution along the length x of the branch of the thermoelectric element when a current density j flows through it has the form

$$\frac{d}{dx}\left(\lambda(x)\frac{dT}{dx}\right)+\frac{j^2}{\sigma}-jT\frac{d\alpha(x)}{dx}=0. \tag{1}$$

The last two terms represent the Joule heat and the distributed Peltier heat, which occurs as a consequence of the density gradient along the branch of the thermoelectric element.

We will assume that the cold side of the branch is adiabatically insulated, and that convective heat transfer occurs between the hot junction and the medium

$$\lambda\left.\frac{dT}{dx}\right|_{x=0}=j\alpha T|_{x=0},$$
$$\lambda\left.\frac{dT}{dx}\right|_{x=1}=j\alpha T|_{x=1}-a(T-\tilde{T})|_{x=1}. \tag{2}$$

We will also assume that the functional dependence on the density of the thermal conductivity λ, the electrical conductivity σ, and the thermoelectric emf α obeys the classical Maxwell statistics for the nondegenerate state of the electric-current carriers [1, 5].

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The set of equations describing the temperature distribution, the heat flow, and the density along the branches of the thermoelectric element can be represented in the normal Cauchy form

$$\dot{q} = -\frac{1}{\lambda}\left(\lambda' q u + \frac{j^2}{\sigma} - j\alpha' T u\right),$$

$$\dot{T} = q, \quad \dot{n} = u, \tag{3}$$

where $\dot{q} = dq/dx$; $\lambda' = d\lambda/dn$, etc.

The variational problem consists in determining the piecewise-smooth function of the current-carrier density gradient u(x), where $x \in [0, 1]$, which together with the solutions T(x) satisfies the given system (2) and (3), and ensures that minimum temperature is obtained at the cold junction

$$T_c = \min T|_{x=0}. \tag{4}$$

This problem will be solved using the Pontryagin maximum principle [6]. It is obvious from physical consideration that no limitations are imposed on the change in density in this case. No appreciable increase in the electrical resistance in the region of the hot junction occurs under finite heat-exchange conditions when its temperature is fixed [4]. The optimum density distribution should automatically ensure a reduction in the parasitic heat flow from the hot to the cold junction.

The Hamilton function, corresponded to the initial system, can be written as follows:

$$H = -\psi_1 \frac{1}{\lambda}\left(\lambda' q u + \frac{j^2}{\sigma} - j\alpha' T u\right) + \psi_2 q + \psi_3 u. \tag{5}$$

The boundary functional can be represented in the form

$$\varphi = T_0 + \nu_1(\lambda_0 q_0 - j\alpha_0 T_0) + \nu_2(\lambda_1 q_1 - j\alpha_1 T_1 + a(T - \tilde{T})), \tag{6}$$

where the subscripts 0 and 1 relate to the cold and hot junctions, respectively.

The conjugate set of equations with corresponding boundary conditions at both ends of the specified range of variation of the independent variable has the form

$$\dot{\psi}_1 = -\frac{\partial H}{\partial q} = \frac{\psi_1}{\lambda}(\lambda u)' - \psi_2,$$

$$\dot{\psi}_2 = -\frac{\partial H}{\partial T} = -\frac{\psi_1}{\lambda} j\alpha' u,$$

$$\dot{\psi}_3 = -\frac{\partial H}{\partial n} = \psi_1\left(-\frac{1}{\lambda^2}\lambda'\left(\lambda' q u + \frac{j^2}{\sigma} - j\alpha' T u\right) + \frac{1}{\lambda}\left(\lambda'' q u + \frac{j^2}{\sigma'} - j\alpha'' T u\right)\right), \tag{7}$$

$$\psi_{10} = \frac{\partial \varphi}{\partial q_0} = \nu_1 \lambda, \quad \psi_{11} = -\frac{\partial \varphi}{\partial q_1} = -\nu_2 \lambda,$$

$$\psi_{20} = \frac{\partial \varphi}{\partial T_0} = 1 - \frac{\psi_{10}}{\lambda}\alpha j, \quad \psi_{21} = -\frac{\partial \varphi}{\partial T_1} = \nu_2(j\alpha - a), \tag{8}$$

$$\psi_{30} = \frac{\partial \varphi}{\partial n_0} = \nu_1(\lambda' q - j\alpha' T), \quad \psi_{31} = -\frac{\partial \varphi}{\partial n_1} = -\nu_2(\lambda' q - j\alpha' T).$$

Taking into account the conditions of the particular mode of operation [6], we obtain the relationship between the conjugate functions inside and on the boundary of the region

$$\psi_1\left(\frac{j^2}{\sigma} + j\alpha' q\right) = -\psi_2(\lambda' q - j\alpha' T). \tag{9}$$

The density gradient, taking into account the conditions of the particular mode of operation, and the Kelly conditions [7]: $\frac{\partial H}{\partial u} = 0,\ \frac{d}{dx}\left(\frac{\partial H}{\partial u}\right)^{\cdot} = 0,\ \frac{d^2}{dx^2}\left(\frac{\partial H}{\partial u}\right) = 0$ can be expressed solely in terms of the initial functions

$$u = j\left[\frac{\alpha' j^2}{\sigma^2 \lambda}(\lambda' q - j\alpha' T) - \left(\frac{j^2}{\sigma'} + j\alpha' q\right)\left(\frac{j}{\sigma'} + 2\alpha' q + \frac{j\lambda'}{\sigma\lambda}\right)\right] \Big/ \left\{\left[\frac{\lambda'}{\lambda}\left(\frac{j}{\sigma'} + j\alpha' q\right) + \left(\frac{j^2}{\sigma''} + j\alpha'' q\right) - \right.\right.$$

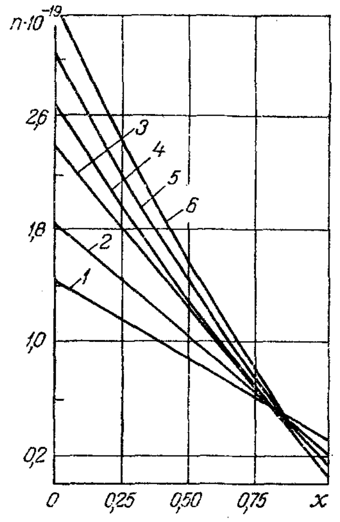

Fig. 1

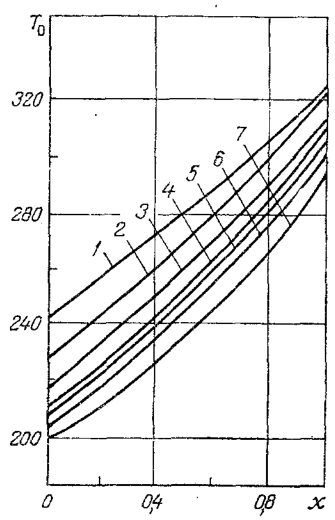

Fig. 2

Fig. 1. Optimum density distribution n (cm^{-3}) along the height x (cm): 1) a = 0.05, 2) 0.1, 3) 0.2, 4) 0.35, 5) 0.5, and 6) 1 W/(cm^2·K).

Fig. 2. Profile of the cold-junction temperature T_0(°K) along the height of the thermoelectric element x (cm): 1) a = 0.05; 2) 0.1; 3) 0.2; 4) 0.35; 5) 0.5; 6) 1 W/(cm^2·K); and 7) a = ∞.

$$-2j\frac{\alpha'}{\lambda}(\lambda' q - j\alpha' T)\Big](\lambda' q - j\alpha' T) - \Big[\lambda'' q - j\alpha'' T - \frac{\lambda'}{\lambda}(\lambda' q - j\alpha' T)\Big]\Big(\frac{j^2}{\sigma'} + j\alpha' q\Big)\Big\}. \tag{10}$$

Using relation (9), the boundary conditions in (8) on the left can also be written solely in terms of the initial functions:

$$\psi_{10} = \frac{\lambda' q - j\alpha' T}{\frac{j\alpha}{\lambda}(\lambda' q - j\alpha' T) - \frac{j^2}{\sigma'} - j\alpha' q},$$
$$\psi_{20} = 1 - \frac{\psi_{10}}{\lambda}\alpha j, \quad \psi_{30} = \frac{\psi_{10}}{\lambda}(\lambda' q - j\alpha' T). \tag{11}$$

To solve the problem expressed by the sets of first-order differential equations (1) and (7) with boundary conditions (2) and (8), and the optimality condition (10), we used the numerical Newton's method [8]. The initial values of the temperature and density are the variable parameters. On the wide boundary when x = 1, there is a discrepancy for the temperature and density, which can be written in terms of the temperature and heat flux. After four-five iterations, with appropriately chosen initial values for the temperature and density, the boundary values at the right end can be satisfied with a specified accuracy.

The calculations were carried out for height d = 1 cm, and T = 293°K, and the coefficient a was specified to have different values in the range from 0.05 to 1 W/(cm^2·°K). In individual vases the values of a in the calculations were taken to be considerably greater in order to compare the results obtained with the results of the solution of the same problem but at a fixed temperature of the hot junction. In view of the invariance of the j·d and a·d transformation, all the results can be recalculated for the corresponding values of the current, heat transfer, and height.

As a result of these calculations, we obtained the optimum current-carrier density distribution along the arms of the thermoelectric element which ensures a maximum drop between the medium and the cold junction. This distribution is characterized by an almost-linear increase in density from the hot to the cold junction. The change in the density along the height of the branch for different heat-transfer coefficients with corresponding current densities are shown in Fig. 1, from which it follows that as the heat transfer increases the carrier density on the cold side increases and that on the hot side decreases.

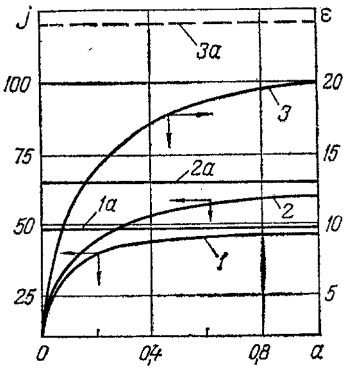

Fig. 3

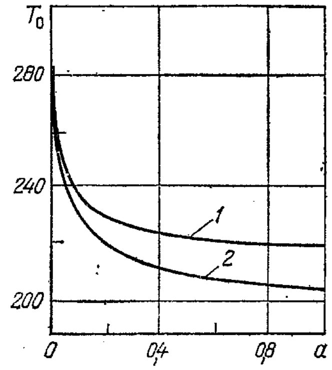

Fig. 4

Fig. 3. Supply current density j (A/cm^2) (curves 1, 1a, 2, and 2a) and relative temperature drop (3, 3a) as a function of the heat-transfer coefficient α ($W/cm^2 \cdot K$): 1) n = const, 2) var, 1a and 2a) values of the current for $\alpha = \infty$, and 3a) value of the relative change in the drop for $\alpha = \infty$.

Fig. 4. Change in the temperature of the cold junction T_0 (K) as a function of the heat-transfer coefficient α ($W/cm^2 \cdot K$): 1) n = const, 2) var.

TABLE 1. Maximum Temperature Drop between the Cold Junction and the Medium for Different Heat-Transfer Coefficients at the Hot Junction

α, W/(cm²·K)	0,05	0,1	0,2	0,35	0,5	1	∞
ΔT_{opt}, K	49;9	64,0	74,9	81,1	83,9	87,8	93,0
ΔT_{cal}, K	46,9	62,9	74,9	80,7	83,3	86,3	
ΔT, K	46,7	58,3	65,9	69,7	71,2	73,1	75,0

Thus for $\alpha = 0.05$ W/(cm²·K) the density which ensures an additional drop of 3°K compared with a uniform distribution decreases from the cold to the hot junction by a factor of 4 from $1.45 \cdot 10^{19}$ to $0.32 \cdot 10^{19}$ cm^{-3}. For a heat-transfer coefficient of 0.5 W/(cm²·K), the additional drop is 12.5°K, and n varies from $3.02 \cdot 10^{19}$ to $6.4 \cdot 10^{17}$ cm^{-3}.

The change in temperature along the height corresponding to the optimum distribution of n(x) for different heat-transfer coefficients is shown in Fig. 2. The figure also shows a graph of the temperature distribution for a nonuniform thermoelectric element at $\alpha \to \infty$, taken from [4]. It can be seen from these graphs that as the heat-transfer increases the temperature distributions approach the curve obtained as $\alpha \to \infty$. The concavity of the curves increases as the heat transfer increases, and it is opposite to the convexity of the temperature-distribution curve for the case n = const. The maximum difference in ΔT which can be achieved for an optimum density distribution compared with n = const for $\alpha = 0.5$ W/(cm²·K) is 12.5°K, and for $\alpha = 1$ W/(cm²·K) it is 15°K, whereas for $\alpha \to \infty$ the theoretical increase in the drop compared with a constant distribution is 18°K. For heat-transfer coefficients which increase from 0.05 to 0.5 W/(cm²·K), the additional drop compared with n = const increases quite sharply, and then gradually approaches its theoretical limit, which is 25% (Fig. 3).

In Fig. 3 we show the optimum current density which gives maximum temperature drop as a function of the heat-transfer coefficients. Compared with a constant distribution n = const the value of the current density shifts to higher values. The dependence of the maximum drop on j has a fairly wide spread, if we choose its optimum distribution n(x) for each value of the current.

In Fig. 4 we show the minimum temperature of the cold junction as a function of the heat-transfer coefficients. It follows from the figure that the change in temperature as the heat transfer increases in the case of a nonuniform thermoelectric element has a smoother form than for n = const. Whereas for n = const with a = 0.25 and 0.5 W/(cm²·K) the drop ΔT differs from its theoretical maximum of 75°K by only 7.5 and 3.5°K (as $a \to \infty$), for the same values of a in a nonuniform thermoelectric element these differences for ΔT amount to 15 and 9°K (ΔT_{opt} = 93°K as $a \to \infty$). The same drop ΔT in a nonuniform thermoelectric element can be achieved for much lower values of the heat-transfer coefficients than in a uniform version. Thus, T_0 = 221.5°K is obtained with a = 0.5 W/(cm²·K) and n = const, and with a = 0.15 W/(cm²·K) for an optimum distribution. As regards the theoretically possible drop when $a \to \infty$ in the uniform case, it is obtained with a = 0.2 W/(cm²·K) in a nonuniform thermoelectric element for similar densities of 45-48 A/cm² in both cases.

The optimum solutions obtained for different values of the heat-transfer coefficient have lower limits of the change in concentration in the region of the hot junction, differing by a factor of ten. At the cold junction for the same heat transfers, the density differs by a factor of 2-2.5. Thus, for a = 0.05 W/(cm²·K) the density at the hot junction $n = 0.32 \cdot 10^{19}$ cm^{-3}, and for a = 1 W/(cm²·K), $n = 0.35 \cdot 10^{18}$ cm^{-3}. At the cold junction, for the same heat transfers, the densities are $1.4 \cdot 10^{19}$ and $3.4 \cdot 10^{19}$ cm^{-3}, respectively.

We can determine how critical the optimum solution, calculated for a specific value of the heat-transfer coefficient, is to the other heat-exchange conditions from the hot junction to the surrounding medium. To do this we will obtain solutions of the heat-conduction equation (1) for a temperature drop ΔT_{cal} for different heat-exchange conditions at the hot junction when the distribution n(x) is optimum with a = 0.2 W/(cm²·K). The results of calculations for the temprature drops for the corresponding optimum current densities are shown in the table. In the last row we have written the values of the temperature drops for n = const. These results show that the distribution of n along the branches is not critical with respect to the value of the heat-transfer coefficient.

NOTATION

T, absolute temperature; ΔT, temperature drop between the cold junction and the medium; q, temperature gradient; n, current-carrier density; u, density gradient; a, heat-transfer coefficient; x, coordinate of the length; d, height of the thermoelectric element; λ, σ, α, thermal conductivity, the electrical conductivity, and the thermoelectric coefficient respectively; and j, current density.

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